

ORTHOGONAL CURVILINEAR COORDINATES

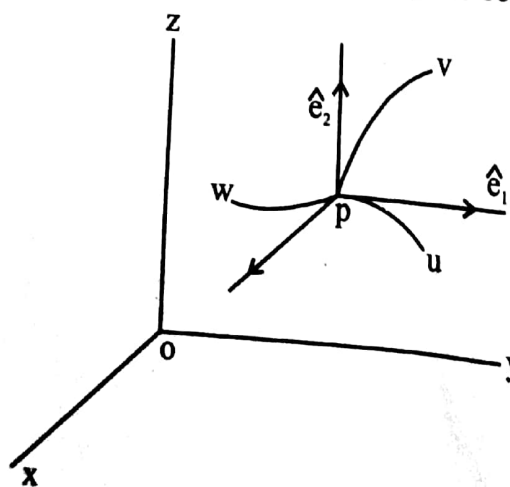
CHAPTER

LEARNING OBJECTIVES

- Meaning of orthogonal curvilinear coordinates. Idea of scale factors or metrical coefficients. Gradient, divergence, curl and Laplacian in terms of orthogonal curvilinear coordinates. Expression for gradient, divergence, curl and Laplacian in cartesian coordinate system.
- Cylindrical coordinates and expression for gradient, divergence, curl and Laplacian in cylindrical coordinates.
- Spherical polar coordinates and expression for gradient, divergence, curl and Laplacian in spherical polar coordinates. Velocity and acceleration in cylindrical coordinate system. Velocity and acceleration in spherical polar coordinate system.

3.1 MEANING ORTHOGONAL CURVILINEAR COORDINATES

Sometimes it is convenient to define the position of a point P in space by three parameters u , v and w which represent three families of surfaces expressed in the form $u = \text{constant}$, $v = \text{constant}$, $w = \text{constant}$. Here u , v and w are continuously differentiable functions. If the three surfaces $u =$



c_1 (constant), $v = c_2$ (constant), $w = c_3$ (constant) intersect in a point P of the region, then values of u , v and w are called curvilinear coordinates of point P . The three surfaces are known as coordinate surfaces. The three surfaces intersect in pairs called coordinate curves. So for u curve, $v = c_2$ (constant), $w = c_3$ (constant). Similarly for v curve, $w = c_3$, $u = c_1$. For w curve, $u = c_1$, $v = c_2$.

The rectangular coordinates (x, y, z) of point P in space is expressed as functions of (u, v, w) . Then $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$.

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or we get the values of u , v and w in term of x , y , z as $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$.

If the coordinate surfaces intersect at right angles at every point $P(x, y, z)$ then the curvilinear coordinate system is called the orthogonal curvilinear system of coordinates.

Let $\hat{e}_1, \hat{e}_2$ and $\hat{e}_3$ be unit vectors along the tangents to the coordinate curves, u, v, w respectively, at point P directed along the increase of u, v and w . So they form right handed triads.

So, $\hat{e}_1 \cdot \hat{e}_2 = \hat{e}_2 \cdot \hat{e}_3 = \hat{e}_1 \cdot \hat{e}_3 = 0$, $\hat{e}_1 \cdot \hat{e}_1 = \hat{e}_2 \cdot \hat{e}_2 = \hat{e}_3 \cdot \hat{e}_3 = 1$, $\hat{e}_2 \times \hat{e}_3 = \hat{e}_1$, $\hat{e}_3 \times \hat{e}_1 = \hat{e}_2$, $\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$

Let us introduce three numbers h_1, h_2 and h_3 known as scale factors or metrical co-

efficients as, $h_1 = \left| \frac{\partial \vec{r}}{\partial u} \right|$, $h_2 = \left| \frac{\partial \vec{r}}{\partial v} \right|$ and $h_3 = \left| \frac{\partial \vec{r}}{\partial w} \right|$

where, $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$.

$$\therefore \frac{\partial \vec{r}}{\partial u} = \hat{i} \frac{\partial x}{\partial u} + \hat{j} \frac{\partial y}{\partial u} + \hat{k} \frac{\partial z}{\partial u}$$

$$\text{So, } h_1 = \left| \frac{\partial \vec{r}}{\partial u} \right| = \sqrt{\left(\frac{\partial x}{\partial u} \right)^2 + \left(\frac{\partial y}{\partial u} \right)^2 + \left(\frac{\partial z}{\partial u} \right)^2}$$

$$\text{Similarly, } h_2 = \left| \frac{\partial \vec{r}}{\partial v} \right| = \sqrt{\left(\frac{\partial x}{\partial v} \right)^2 + \left(\frac{\partial y}{\partial v} \right)^2 + \left(\frac{\partial z}{\partial v} \right)^2}$$

$$\text{and } h_3 = \left| \frac{\partial \vec{r}}{\partial w} \right| = \sqrt{\left(\frac{\partial x}{\partial w} \right)^2 + \left(\frac{\partial y}{\partial w} \right)^2 + \left(\frac{\partial z}{\partial w} \right)^2}$$

The tangent vector to u curve at point P is $\frac{\partial \vec{r}}{\partial u}$, But $\hat{e}_1$ is the unit vector in this direction.

$$\text{Then } \frac{\partial \vec{r}}{\partial u} = \left| \frac{\partial \vec{r}}{\partial u} \right| \hat{e}_1 = h_1 \hat{e}_1$$

$$\text{Similarly, } \frac{\partial \vec{r}}{\partial v} = h_2 \hat{e}_2 \text{ and } \frac{\partial \vec{r}}{\partial w} = h_3 \hat{e}_3$$

Gradient in terms of Orthogonal Curvilinear Coordinates :

Let $\phi(u, v, w)$ be any scalar point Function, and $\hat{e}_1, \hat{e}_2$ and $\hat{e}_3$ are the mutually orthogonal unit vectors along the tangents to the coordinate curves u, v and w respectively.

So we write $\vec{\nabla} \phi = \phi_1 \hat{e}_1 + \phi_2 \hat{e}_2 + \phi_3 \hat{e}_3$
where, ϕ_1 , ϕ_2 and ϕ_3 are to be determined.

The position vector is $\vec{r} = \vec{r}(u, v, w)$.

$$\therefore d\vec{r} = \frac{\partial \vec{r}}{\partial u} du + \frac{\partial \vec{r}}{\partial v} dv + \frac{\partial \vec{r}}{\partial w} dw$$

$$= h_1 \hat{e}_1 du + h_2 \hat{e}_2 dv + h_3 \hat{e}_3 dw$$

$$\therefore \vec{\nabla} \phi \cdot d\vec{r} = h_1 \phi_1 du + h_2 \phi_2 dv + h_3 \phi_3 dw$$

$$\text{or } d\phi = h_1 \phi_1 du + h_2 \phi_2 dv + h_3 \phi_3 dw$$

$$\text{But the differential is } d\phi = \frac{\partial \phi}{\partial u} du + \frac{\partial \phi}{\partial v} dv + \frac{\partial \phi}{\partial w} dw$$

Hence from equation (3) and (4) we get,

$$h_1 \phi_1 = \frac{\partial \phi}{\partial u} \Rightarrow \boxed{\phi_1 = \frac{1}{h_1} \frac{\partial \phi}{\partial u}}$$

$$h_2 \phi_2 = \frac{\partial \phi}{\partial v} \Rightarrow \boxed{\phi_2 = \frac{1}{h_2} \frac{\partial \phi}{\partial v}}$$

$$h_3 \phi_3 = \frac{\partial \phi}{\partial w} \Rightarrow \boxed{\phi_3 = \frac{1}{h_3} \frac{\partial \phi}{\partial w}}$$

Putting these values in equation (1) we get,

$$\vec{\nabla} \phi = \frac{1}{h_1} \frac{\partial \phi}{\partial u} \hat{e}_1 + \frac{1}{h_2} \frac{\partial \phi}{\partial v} \hat{e}_2 + \frac{1}{h_3} \frac{\partial \phi}{\partial w} \hat{e}_3$$

which is the required expression,

$$\text{Note - If } \phi = u \text{ then, } \vec{\nabla} \phi = \vec{\nabla} u = \frac{1}{h_1} \frac{\partial \phi}{\partial u} \hat{e}_1 = \hat{e}_1 / h_1$$

$$\text{Similarly, } \vec{\nabla} v = \hat{e}_2 / h_2, \vec{\nabla} w = \hat{e}_3 / h_3$$

Divergence in terms of Orthogonal Curvilinear Coordinates :

Let $\vec{A}(u, v, w)$ be vector point function having components A_1 , A_2 and A_3 along unit vectors $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ respectively.

$$\text{So, } \vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 = A_1 (\hat{e}_2 \times \hat{e}_3) + A_2 (\hat{e}_3 \times \hat{e}_1) + A_3 (\hat{e}_1 \times \hat{e}_2)$$

$$\text{or } \vec{A} = A_1 h_2 h_3 \vec{\nabla} v \times \vec{\nabla} w + A_2 h_1 h_3 \vec{\nabla} w \times \vec{\nabla} u + A_3 h_1 h_2 \vec{\nabla} u \times \vec{\nabla} v$$

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$$\left\{ \text{using } \hat{e}_1 = h_1 \vec{\nabla} u, \hat{e}_2 = h_2 \vec{\nabla} v, \hat{e}_3 = h_3 \vec{\nabla} w \right\}$$

$$\text{So, } \vec{\nabla} \cdot \vec{A} = \vec{\nabla} \cdot \left(A_1 h_2 h_3 \vec{\nabla} v \times \vec{\nabla} w + A_2 h_1 h_3 \vec{\nabla} w \times \vec{\nabla} u + A_3 h_1 h_2 \vec{\nabla} u \times \vec{\nabla} v \right)$$

$$\text{or } \vec{\nabla} \cdot \vec{A} = \vec{\nabla} \cdot \left(A_1 h_2 h_3 \vec{\nabla} v \times \vec{\nabla} w \right) + \vec{\nabla} \cdot \left(A_2 h_1 h_3 \vec{\nabla} w \times \vec{\nabla} u \right) + \vec{\nabla} \cdot \left(A_3 h_1 h_2 \vec{\nabla} u \times \vec{\nabla} v \right) \quad \dots(1)$$

$$\text{But from the vector identity, } \vec{\nabla} \cdot (\phi \vec{F}) = \phi \vec{\nabla} \cdot \vec{F} + \vec{\nabla} \phi \cdot \vec{F}$$

$$\text{So, } \vec{\nabla} \cdot \left(A_1 h_2 h_3 \vec{\nabla} v \times \vec{\nabla} w \right) = A_1 h_2 h_3 \vec{\nabla} \cdot \left(\vec{\nabla} v \times \vec{\nabla} w \right) + \vec{\nabla} (A_1 h_2 h_3) \cdot \vec{\nabla} v \times \vec{\nabla} w \quad \dots(2)$$

$$\text{Again we use the identity, } \vec{\nabla} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot \vec{\nabla} \times \vec{B} - \vec{B} \cdot \vec{\nabla} \times \vec{C}$$

$$\text{Then, } \vec{\nabla} \cdot \left(\vec{\nabla} v \times \vec{\nabla} w \right) = \vec{\nabla} w \cdot \vec{\nabla} \times \vec{\nabla} v - \vec{\nabla} v \cdot \vec{\nabla} \times \vec{\nabla} w$$

$$= 0 \quad \text{as } \vec{\nabla} \times \vec{\nabla} v = 0, \vec{\nabla} \times \vec{\nabla} w = 0$$

$$\text{Then equation (2) becomes, } \vec{\nabla} \cdot \left(A_1 h_2 h_3 \vec{\nabla} v \times \vec{\nabla} w \right) = \vec{\nabla} (A_1 h_2 h_3) \cdot \vec{\nabla} v \times \vec{\nabla} w$$

$$\text{or } \vec{\nabla} \cdot \left(A_1 h_2 h_3 \vec{\nabla} v \times \vec{\nabla} w \right) = \left(\vec{\nabla} v \times \vec{\nabla} w \right) \cdot \vec{\nabla} (A_1 h_2 h_3)$$

$$= \vec{\nabla} v \times \vec{\nabla} w \cdot \left[\frac{\partial}{\partial u} (A_1 h_2 h_3) \vec{\nabla} u + \frac{\partial}{\partial v} (A_1 h_2 h_3) \vec{\nabla} v + \frac{\partial}{\partial w} (A_1 h_2 h_3) \vec{\nabla} w \right] \quad \dots(1)$$

$$= \vec{\nabla} v \times \vec{\nabla} w \cdot \vec{\nabla} u \cdot \frac{\partial}{\partial u} (A_1 h_2 h_3) \quad \text{as } \vec{\nabla} v \times \vec{\nabla} w \cdot \vec{\nabla} v = 0, \vec{\nabla} v \times \vec{\nabla} w \cdot \vec{\nabla} w = 0$$

$$= \frac{\hat{e}_2}{h_2} \times \frac{\hat{e}_3}{h_3} \cdot \frac{\hat{e}_1}{h_1} \frac{\partial}{\partial u} (A_1 h_2 h_3) \quad \text{using } \vec{\nabla} u = \hat{e}_1 / h_1, \vec{\nabla} v = \hat{e}_2 / h_2, \vec{\nabla} w = \hat{e}_3 / h_3$$

$$= \frac{\hat{e}_1 \cdot \hat{e}_1}{h_1 h_2 h_3} \frac{\partial}{\partial u} (A_1 h_2 h_3) \quad \left\{ \because \hat{e}_2 \times \hat{e}_3 = \hat{e}_1 \right\}$$

$$\text{i.e., } \vec{\nabla} \cdot \left(A_1 h_2 h_3 \vec{\nabla} v \times \vec{\nabla} w \right) = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u} (A_1 h_2 h_3) \quad \dots(2)$$

$$\text{Similarly, } \vec{\nabla} \cdot \left(A_2 h_3 h_1 \vec{\nabla} w \times \vec{\nabla} u \right) = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial v} (A_2 h_3 h_1) \quad \dots(3)$$

$$\text{and } \vec{\nabla} \cdot (A_3 h_1 h_2 \vec{\nabla} u \times \vec{\nabla} v) = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial w} (A_3 h_1 h_2) \quad \dots(4)$$

Putting these values in equation (1) we get,

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u} (A_1 h_2 h_3) + \frac{\partial}{\partial v} (A_2 h_3 h_1) + \frac{\partial}{\partial w} (A_3 h_1 h_2) \right]$$

which is the required expression.

Curl in terms of Orthogonal Curvilinear Coordinates :

Let $\vec{A}(u, v, w)$ be vector point function given in terms of Orthogonal Curvilinear Coordinates, u, v and w . If A_1, A_2 and A_3 be components of $\vec{A}$ along $\hat{e}_1, \hat{e}_2$ and $\hat{e}_3$ the unit vectors along u, v and w axes then, $\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$.

$$= h_1 A_1 \vec{\nabla} u + h_2 A_2 \vec{\nabla} v + h_3 A_3 \vec{\nabla} w \quad \text{as } \hat{e}_1 = h_1 \vec{\nabla} u, \hat{e}_2 = h_2 \vec{\nabla} v, \hat{e}_3 = h_3 \vec{\nabla} w$$

$$\text{So, } \vec{\nabla} \times \vec{A} = \vec{\nabla} \times [h_1 A_1 \vec{\nabla} u + h_2 A_2 \vec{\nabla} v + h_3 A_3 \vec{\nabla} w]$$

$$\text{i.e., } (\vec{\nabla} \times \vec{A}) = \vec{\nabla} \times (h_1 A_1 \vec{\nabla} u) + \vec{\nabla} \times (h_2 A_2 \vec{\nabla} v) + \vec{\nabla} \times (h_3 A_3 \vec{\nabla} w) \quad \dots(1)$$

$$\text{From vector identity } \vec{\nabla} \times (\phi \vec{F}) = \phi \vec{\nabla} \times \vec{F} + \vec{\nabla} \phi \times \vec{F}$$

$$\text{Then, } \vec{\nabla} \times (h_1 A_1 \vec{\nabla} u) = h_1 A_1 \vec{\nabla} \times \vec{\nabla} u + \vec{\nabla} (h_1 A_1) \times \vec{\nabla} u$$

$$= \vec{\nabla} (h_1 A_1) \times \vec{\nabla} u \quad \text{as } \vec{\nabla} \times \vec{\nabla} u = 0$$

$$\text{i.e., } \vec{\nabla} \times (h_1 A_1 \vec{\nabla} u) = \left[\frac{\partial}{\partial u} (h_1 A_1) \vec{\nabla} u + \frac{\partial}{\partial v} (h_1 A_1) \vec{\nabla} v + \frac{\partial}{\partial w} (h_1 A_1) \vec{\nabla} w \right] \times \vec{\nabla} u$$

$$\text{or, } \vec{\nabla} \times (h_1 A_1 \vec{\nabla} u) = \frac{\partial}{\partial v} (h_1 A_1) \vec{\nabla} v \times \vec{\nabla} u + \frac{\partial}{\partial w} (h_1 A_1) \vec{\nabla} w \times \vec{\nabla} u \quad \text{as } \vec{\nabla} u \times \vec{\nabla} u = 0$$

$$= \frac{\partial}{\partial v} (h_1 A_1) \frac{\hat{e}_2}{h_2} \times \frac{\hat{e}_1}{h_1} + \frac{\partial}{\partial w} (h_1 A_1) \frac{\hat{e}_3}{h_3} \times \frac{\hat{e}_1}{h_1}$$

$$\left\{ \text{using } \vec{\nabla} u = \frac{\hat{e}_1}{h_1}, \vec{\nabla} v = \frac{\hat{e}_2}{h_2}, \vec{\nabla} w = \frac{\hat{e}_3}{h_3} \right\}$$

$$\text{or } \vec{\nabla} \times (h_1 A_1 \vec{\nabla} u) = -\frac{1}{h_1 h_2} \frac{\partial}{\partial v} (h_1 A_1) \hat{e}_3 + \frac{1}{h_3 h_1} \frac{\partial}{\partial w} (h_1 A_1) \hat{e}_2 \quad \dots(2)$$

Similarly we have, $\vec{\nabla} \times (h_2 A_2 \vec{\nabla} v) = -\frac{1}{h_2 h_3} \frac{\partial}{\partial w} (h_2 A_2) \hat{e}_1 + \frac{1}{h_1 h_2} \frac{\partial}{\partial u} (h_2 A_2) \hat{e}_3$ (3)

$$\vec{\nabla} \times (h_3 A_3 \vec{\nabla} w) = -\frac{\hat{e}_2}{h_3 h_1} \frac{\partial}{\partial u} (h_3 A_3) + \frac{\hat{e}_1}{h_2 h_3} \frac{\partial}{\partial v} (h_3 A_3)$$
(4)

Substituting equations (2), (3) and (4) in equation (1) we get,

$$\begin{aligned} \vec{\nabla} \times \vec{A} = & \frac{1}{h_2 h_3} \left[\frac{\partial}{\partial v} (h_3 A_3) - \frac{\partial}{\partial w} (h_2 A_2) \right] \hat{e}_1 \\ & + \frac{1}{h_3 h_1} \left[\frac{\partial}{\partial w} (h_1 A_1) - \frac{\partial}{\partial u} (h_3 A_3) \right] \hat{e}_2 + \frac{1}{h_1 h_2} \left[\frac{\partial}{\partial u} (h_2 A_2) - \frac{\partial}{\partial v} (h_1 A_1) \right] \hat{e}_3 \end{aligned}$$

$$\text{or } \vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$
(5)

Laplacian operator (∇^2) in terms of Orthogonal curvilinear coordinates.

Laplacian operator is given by $\nabla^2 = \vec{\nabla} \cdot \vec{\nabla}$

$$\therefore \nabla^2 \phi = \vec{\nabla} \cdot \vec{\nabla} \phi$$
(1)

But if we consider $\vec{A} = \vec{\nabla} \phi$

$$\text{then } \nabla^2 \phi = \vec{\nabla} \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u} (A_1 h_2 h_3) + \frac{\partial}{\partial v} (A_2 h_3 h_1) + \frac{\partial}{\partial w} (A_3 h_1 h_2) \right]$$
(2)

in term of orthogonal curvilinear coordinates,

$$\text{Here } \vec{A} = \vec{\nabla} \phi = \frac{1}{h_1} \frac{\partial \phi}{\partial u} \hat{e}_1 + \frac{1}{h_2} \frac{\partial \phi}{\partial v} \hat{e}_2 + \frac{1}{h_3} \frac{\partial \phi}{\partial w} \hat{e}_3 = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$$

$$\Rightarrow A_1 = \frac{1}{h_1} \frac{\partial \phi}{\partial u}, A_2 = \frac{1}{h_2} \frac{\partial \phi}{\partial v}, A_3 = \frac{1}{h_3} \frac{\partial \phi}{\partial w}$$

Equation (2) therefore becomes,

$$\begin{aligned} \nabla^2 \phi = & \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u} \left(\frac{1}{h_1} \frac{\partial \phi}{\partial u} h_2 h_3 \right) + \frac{\partial}{\partial v} \left(\frac{1}{h_2} \frac{\partial \phi}{\partial v} h_3 h_1 \right) + \frac{\partial}{\partial w} \left(\frac{1}{h_3} \frac{\partial \phi}{\partial w} h_1 h_2 \right) \right] \\ \text{i.e., } \nabla^2 \phi = & \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u} \left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{h_3 h_1}{h_2} \frac{\partial \phi}{\partial v} \right) + \frac{\partial}{\partial w} \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial w} \right) \right] \end{aligned}$$
(3)

which is the required expression.

Expression for gradient, divergence, curl and laplacian operator in cartesian coordinate system (Rectangular coordinates).

For Cartesian coordinate system or rectangular coordinate system we use transformation equations, $u = x, v = y, w = z$.

$$\text{Then, } h_1 = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2} \text{ becomes } h_1 = 1 \text{ as } \frac{\partial x}{\partial x} = 1, \frac{\partial y}{\partial x} = 0, \frac{\partial z}{\partial x} = 0$$

Similarly, $h_2 = 1, h_3 = 1$.

The expression for gradient in orthogonal curvilinear coordinates is given by,

$$\nabla^2 \phi = \frac{1}{h_1} \frac{\partial \phi}{\partial u} \hat{e}_1 + \frac{1}{h_2} \frac{\partial \phi}{\partial v} \hat{e}_2 + \frac{1}{h_3} \frac{\partial \phi}{\partial w} \hat{e}_3$$

So in Rectangular coordinate system,

$$u = x, v = y, w = z, h_1 = h_2 = h_3 = 1$$

$\hat{e}_1 = \hat{i}$ along x axis $\hat{e}_2 = \hat{j}$ along y - axis and $\hat{e}_3 = \hat{k}$ along z axis.

$$\text{So, } \vec{\nabla} \phi = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}$$

Similarly divergence of vector $\vec{A}$ is represented in the orthogonal curvilinear system as

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u} (A_1 h_2 h_3) + \frac{\partial}{\partial v} (A_2 h_3 h_1) + \frac{\partial}{\partial w} (A_3 h_1 h_2) \right]$$

$$\text{or } \vec{\nabla} \cdot \vec{A} = \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right)$$

Also curl of vector function in orthogonal curvilinear coordinate system is given by,

$$\vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix}$$

Putting $h_1 = h_2 = h_3 = 1, u = x, v = y, w = z, \hat{e}_1 = \hat{i}, \hat{e}_2 = \hat{j}$ and $\hat{e}_3 = \hat{k}$ we get,

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

Finally expression for laplacian in orthogonal curvilinear coordinate system is given by,

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$$\nabla^2 \phi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u} \left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{h_3 h_1}{h_2} \frac{\partial \phi}{\partial v} \right) + \frac{\partial}{\partial w} \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial w} \right) \right]$$

For cartesian coordinate system we get, $\nabla^2 \phi = \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial \phi}{\partial z} \right)$

$$\text{or } \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

3.2 CYLINDRICAL COORDINATES

In the circular cylindrical coordinate system the three curvilinear coordinates (u , v and w) are specified by (r, θ, z) , such that it consists of (a) Right circular cylinder with Z-axis as common axis with $r = \sqrt{x^2 + y^2} = \text{constant}$.

(b) Half planes through Z-axis with

$$\theta = \tan^{-1} \frac{y}{x} = \text{constant}.$$

(c) Planes parallel to xy plane, $z = \text{constant}$.

Figure shows a point P in space with cartesian coordinates (x, y, z) .

$r = O'P$ is magnitude of radius vector PM is parallel to Z-axis, with OM parallel to O'P $\angle MOX = \theta$.

In the following figure shown, consider ΔOSQ ($\angle OSQ = 90^\circ$).

$$\cos \theta = \frac{OS}{OQ} \Rightarrow OS = OQ \cos \theta$$

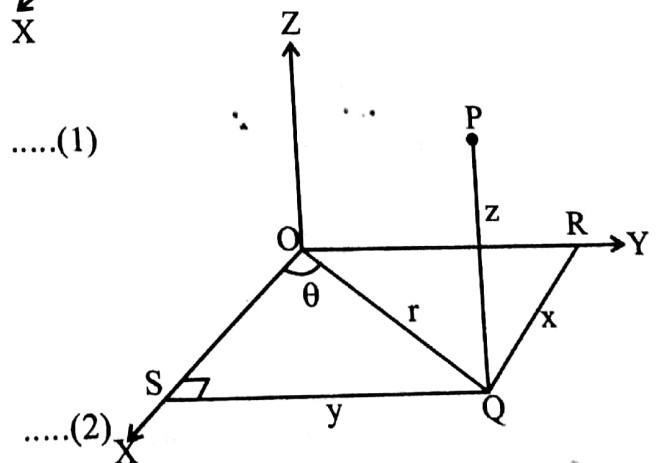
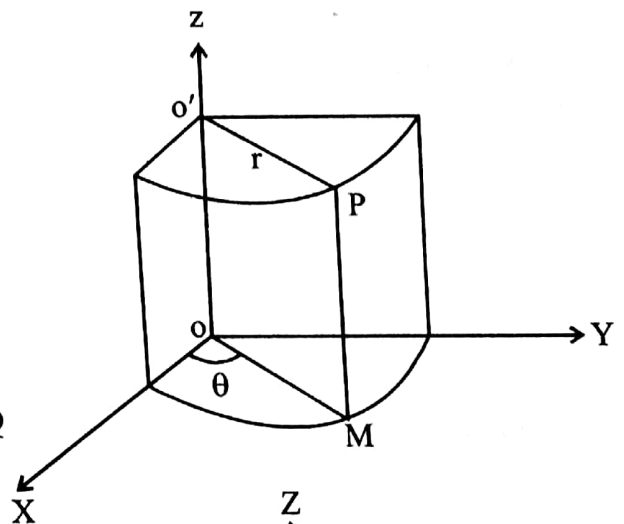
$$\Rightarrow \boxed{x = r \cos \theta}$$

$$\text{Also } \sin \theta = \frac{SQ}{OQ}$$

$$\text{or } \sin \theta = \frac{y}{r}$$

$$\Rightarrow \boxed{y = r \sin \theta}$$

$$\text{and } \boxed{z = z}$$



.....(1)

.....(2)

.....(3)

Equations (1), (2) and (3) are transformation equations between rectangular coordinates (x, y, z) and cylindrical coordinates (r, θ, z) .

So for cylindrical coordinates $u = r, v = \theta, w = z$.

$$\hat{e}_1 = \hat{e}_r, \hat{e}_2 = \hat{e}_\theta, \hat{e}_3 = \hat{e}_z$$

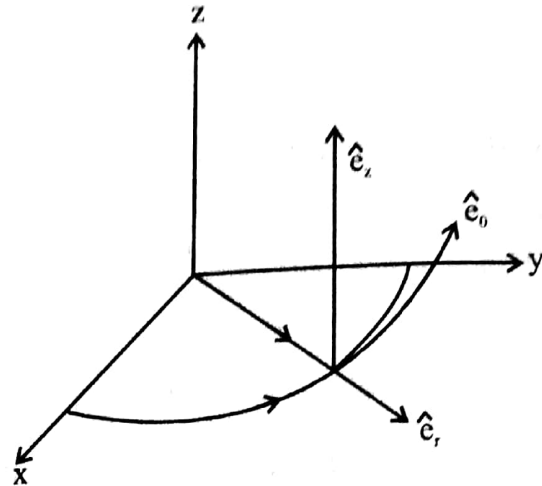
The scale factors are found as follows.

$$h_1 = h_r = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2} \\ = \sqrt{(\cos \theta)^2 + (\sin \theta)^2 + 0} = 1.$$

$$h_2 = h_\theta = \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} \\ = \sqrt{(-r \sin \theta)^2 + (r \cos \theta)^2 + 0}$$

$$\text{or } h_2 = h_\theta = r$$

$$h_3 = h_z = \sqrt{\left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2} = 1.$$



Now we are able to find expressions for various differential operators in cylindrical coordinates as discussed below.

Gradient : The expression for gradient in orthogonal curvilinear coordinates is given as,

$$\vec{\nabla} \phi = \frac{\hat{e}_1}{h_1} \frac{\partial \phi}{\partial u} + \frac{\hat{e}_2}{h_2} \frac{\partial \phi}{\partial v} + \frac{\hat{e}_3}{h_3} \frac{\partial \phi}{\partial w}$$

So, using $u = r$, $v = \theta$, $w = z$ and putting values of h_1 , h_2 and h_3 .

$$\text{we get, } \vec{\nabla} \phi = \frac{\partial \phi}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{e}_\theta + \frac{\partial \phi}{\partial z} \hat{e}_z \quad \text{.....(4)}$$

Divergence :

The expression for divergence in orthogonal curvilinear coordinates is,

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u} (A_1 h_2 h_3) + \frac{\partial}{\partial v} (A_2 h_3 h_1) + \frac{\partial}{\partial w} (A_3 h_1 h_2) \right]$$

So in cylindrical coordinate system we have,

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r} \left[\frac{\partial}{\partial r} (A_1 r) + \frac{\partial}{\partial \theta} (A_2) + \frac{\partial}{\partial z} (A_3 r) \right]$$

$$\text{or } \vec{\nabla} \cdot \vec{A} = \frac{1}{r} \frac{\partial}{\partial r} (A_1 r) + \frac{1}{r} \frac{\partial A_2}{\partial \theta} + \frac{\partial A_3}{\partial z}$$

.....(5)

Curl : In orthogonal curvilinear coordinates

$$\vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

Then in cylindrical coordinates

$$\vec{\nabla} \times \vec{A} = \frac{1}{r} \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_1 & rA_2 & A_3 \end{vmatrix}$$

$$\text{or } \vec{\nabla} \times \vec{A} = \left(\frac{1}{r} \frac{\partial A_3}{\partial \theta} - \frac{\partial A_2}{\partial z} \right) \hat{e}_r + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial r} \right) \hat{e}_\theta + \frac{1}{r} \left(\frac{\partial}{\partial r} (rA_2) - \frac{\partial A_1}{\partial \theta} \right) \hat{e}_z \quad \dots\dots(6)$$

Laplacian : In orthogonal curvilinear coordinates

$$\nabla^2 \phi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u} \left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{h_3 h_1}{h_2} \frac{\partial \phi}{\partial v} \right) + \frac{\partial}{\partial w} \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial w} \right) \right]$$

So, in term of cylindrical coordinates,

$$\nabla^2 \phi = \frac{1}{r} \left[\frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial \phi}{\partial \theta} \right) + \frac{\partial}{\partial z} \left(r \frac{\partial \phi}{\partial z} \right) \right]$$

$$= \frac{1}{r} \left[r \frac{\partial^2 \phi}{\partial r^2} + \frac{\partial \phi}{\partial r} + \frac{1}{r} \frac{\partial^2 \phi}{\partial \theta^2} + r \frac{\partial^2 \phi}{\partial z^2} \right]$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} \quad \dots\dots(7)$$

3.3 SPHERICAL POLAR COORDINATES

Let P (x, y, z) be any point in space. The spherical polar coordinates of point P are given by $u = r$, $v = \theta$, $w = \phi$, where $r = OP$ is distance of point P from the origin.

θ is the angle which OP makes with Z-axis. ϕ is the angle between OQ and X-axis.

In ΔOPR ; $\angle ORP = 90^\circ$

$$\cos \theta = \frac{OR}{OP} \Rightarrow OR = OP \cos \theta$$

$$\Rightarrow \boxed{z = r \cos \theta} \quad \dots\dots(1)$$

$$\sin \theta = \frac{PR}{OP} \Rightarrow PR = OP \sin \theta$$

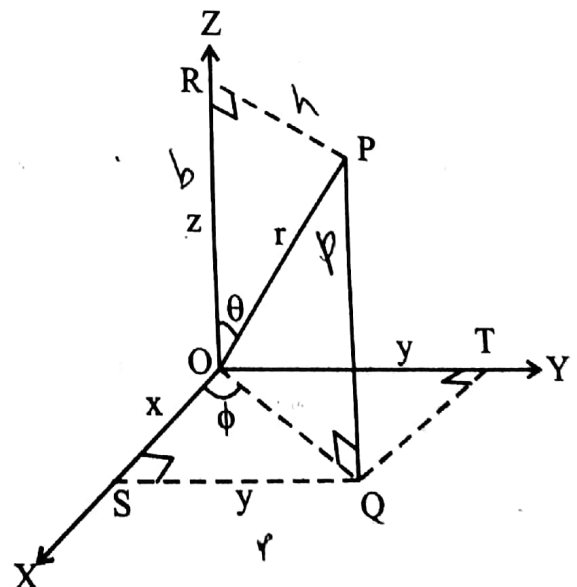
$$\Rightarrow PR = r \sin \theta.$$

$$\Rightarrow OQ = r \sin \theta. \quad \dots\dots(2)$$

Also in ΔOSQ ; $\angle OSQ = 90^\circ$

$$\text{Then } \cos \phi = \frac{OS}{OQ} \Rightarrow OS = OQ \cos \phi$$

$$\Rightarrow \boxed{x = r \sin \theta \cos \phi} \quad \dots\dots(3)$$



Again, $\sin \phi = \frac{SQ}{OQ} \Rightarrow SQ = OQ \sin \phi \Rightarrow OT = OQ \sin \phi$ (4)

i.e., $y = r \sin \theta \sin \phi$

Hence the transformation equations are given by,

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

Hence $x^2 + y^2 + z^2 = r^2$ i.e., $r = \sqrt{x^2 + y^2 + z^2} = \text{constant}$ gives family of spheres with centre at origin,

Also $\theta = \cos^{-1} \frac{z}{r} = \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \text{constant}$ describes the right circular cones with

vertex at the origin.

And $\phi = \tan^{-1} \left(\frac{y}{x} \right) = \text{constant}$ gives half planes through the polar axis i.e., z-axis.

So, in this case $u = r, v = \theta, w = \phi$ and $\hat{e}_1 = \hat{e}_r, \hat{e}_2 = \hat{e}_\theta, \hat{e}_3 = \hat{e}_\phi$

From transformation equations,

$$\frac{\partial x}{\partial r} = \sin \theta \cos \phi \quad \frac{\partial x}{\partial \theta} = r \cos \theta \cos \phi \quad \frac{\partial x}{\partial \phi} = -r \sin \theta \sin \phi$$

$$\frac{\partial y}{\partial r} = \sin \theta \sin \phi \quad \frac{\partial y}{\partial \theta} = r \cos \theta \sin \phi \quad \frac{\partial y}{\partial \phi} = r \sin \theta \cos \phi$$

$$\frac{\partial z}{\partial r} = \cos \theta \quad \frac{\partial z}{\partial \theta} = -r \sin \theta \quad \frac{\partial z}{\partial \phi} = 0.$$

Then scale factors are given by,

$$h_1^2 = \left(\frac{\partial x}{\partial r} \right)^2 + \left(\frac{\partial y}{\partial r} \right)^2 + \left(\frac{\partial z}{\partial r} \right)^2 = \sin^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi + \cos^2 \theta$$

i.e., $h_1^2 = 1$ or $h_1 = 1$

$$h_2^2 = \left(\frac{\partial x}{\partial \theta} \right)^2 + \left(\frac{\partial y}{\partial \theta} \right)^2 + \left(\frac{\partial z}{\partial \theta} \right)^2 = r^2 \cos^2 \theta \cos^2 \phi + r^2 \cos^2 \theta \sin^2 \phi + r^2 \sin^2 \theta$$

$$\Rightarrow h_2^2 = r^2 \Rightarrow h_2 = r$$

$$h_3^2 = \left(\frac{\partial x}{\partial \phi} \right)^2 + \left(\frac{\partial y}{\partial \phi} \right)^2 + \left(\frac{\partial z}{\partial \phi} \right)^2 = r^2 \sin^2 \theta \sin^2 \phi + r^2 \sin^2 \theta \cos^2 \phi$$

Orthogonal Curvilinear Coordinates

$$\text{or, } h_3^2 = r^2 \sin^2 \theta \Rightarrow \boxed{h_3 = r \sin \theta}$$

Differential operators in spherical polar coordinates are found out as described below.

Gradient : $\vec{\nabla} f = \frac{\hat{e}_1}{h_1} \frac{\partial f}{\partial u} + \frac{\hat{e}_2}{h_2} \frac{\partial f}{\partial v} + \frac{\hat{e}_3}{h_3} \frac{\partial f}{\partial w}$ in curvilinear coordinates.

using $u = r, v = \theta, w = \phi, h_1 = 1, h_2 = r, h_3 = r \sin \theta$.

$$\text{i.e., } \vec{\nabla} f = \hat{e}_r \frac{\partial f}{\partial r} + \frac{\hat{e}_\theta}{r} \frac{\partial f}{\partial \theta} + \frac{\hat{e}_\phi}{r \sin \theta} \frac{\partial f}{\partial \phi} \quad \dots(5)$$

$$\text{i.e., } \vec{\nabla} = \hat{e}_r \frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r} \frac{\partial}{\partial \theta} + \frac{\hat{e}_\phi}{r \sin \theta} \frac{\partial}{\partial \phi}$$

Divergence : $\vec{\nabla} \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u} (h_2 h_3 A_1) + \frac{\partial}{\partial v} (h_3 h_1 A_2) + \frac{\partial}{\partial w} (h_1 h_2 A_3) \right]$

in curvilinear coordinates which in spherical polar coordinates becomes

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r} (r^2 \sin \theta A_1) + \frac{\partial}{\partial \theta} (r \sin \theta A_2) + \frac{\partial}{\partial \phi} (r A_3) \right]$$

$$\text{i.e., } \vec{\nabla} \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_1) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_2) + \frac{1}{r \sin \theta} \frac{\partial A_3}{\partial \phi} \quad \dots(6)$$

Curl : $\vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix}$

becomes $\vec{\nabla} \times \vec{A} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{e}_r & r \hat{e}_\theta & r \sin \theta \hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_1 & r A_2 & r \sin \theta A_3 \end{vmatrix}$

$$\text{i.e., } \vec{\nabla} \times \vec{A} = \frac{1}{r^2 \sin \theta}$$

$$\left[\left\{ \frac{\partial}{\partial \theta} (r \sin \theta A_3) - \frac{\partial}{\partial \phi} (r A_2) \right\} \hat{e}_r - \left\{ \frac{\partial}{\partial r} (r \sin \theta A_3) - \frac{\partial}{\partial \phi} A_1 \right\} r \hat{e}_\theta + \left\{ \frac{\partial}{\partial r} (r A_2) - \frac{\partial A_1}{\partial \theta} \right\} r \sin \theta \hat{e}_\phi \right] \quad \dots(7)$$

Laplacian : In term of orthogonal curvilinear coordinates

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u} \left(\frac{h_2 h_3}{h_1} \frac{\partial f}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{h_3 h_1}{h_2} \frac{\partial f}{\partial v} \right) + \frac{\partial}{\partial w} \left(\frac{h_1 h_2}{h_3} \frac{\partial f}{\partial w} \right) \right]$$

∴ In spherical polar coordinates we get,

$$\begin{aligned}\nabla^2 f &= \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r} \left(r^2 \sin \theta \frac{\partial f}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{\partial}{\partial \phi} \left(\frac{1}{\sin \theta} \frac{\partial f}{\partial \phi} \right) \right] \\ &= \frac{1}{r^2 \sin \theta} \left[\sin \theta \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 f}{\partial \phi^2} \right] \\ \text{or } \nabla^2 f &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} \quad \dots (8)\end{aligned}$$

Velocity and Acceleration In cylindrical coordinates

The position vector $\vec{r}$ in cylindrical coordinate is $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$

where $x = r \cos \theta$ $y = r \sin \theta$ $z = z$.

∴ $\vec{r} = \hat{i}r \cos \theta + \hat{j}r \sin \theta + \hat{k}z$ (1) the unit vectors $\hat{e}_r, \hat{e}_\theta$ are given by

$$\hat{e}_r = \frac{\frac{\partial \vec{r}}{\partial r}}{\left| \frac{\partial \vec{r}}{\partial r} \right|} = \cos \theta \hat{i} + \sin \theta \hat{j}, \quad \hat{e}_\theta = \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left| \frac{\partial \vec{r}}{\partial \theta} \right|} = -\sin \theta \hat{i} + \cos \theta \hat{j}$$

So equation (1) becomes $\vec{r} = r(\hat{i} \cos \theta + \hat{j} \sin \theta) + \hat{k}z$

$$\Rightarrow \vec{r} = r\hat{e}_r + \hat{k}z \quad \dots (2)$$

So velocity in cylindrical coordinate is given by $\Rightarrow \vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{e}_r + \hat{k}z)$

$$\text{i.e. } \vec{v} = \hat{e}_r \frac{dr}{dt} + r \frac{d}{dt}(\hat{e}_r) + \hat{k} \frac{dz}{dt} \quad \dots (3)$$

$$\begin{aligned}\text{But } \frac{d}{dt}(\hat{e}_r) &= \frac{d}{dt}[\cos \theta \hat{i} + \sin \theta \hat{j}] = -\sin \theta \dot{\theta} \hat{i} + \cos \theta \dot{\theta} \hat{j} \text{ where } \dot{\theta} = \frac{d\theta}{dt} \\ &= \dot{\theta}[-\sin \theta \hat{i} + \cos \theta \hat{j}]\end{aligned}$$

$$\text{or } \frac{d}{dt}(\hat{e}_r) = \dot{\theta} \hat{e}_\theta \quad \dots (4) \text{ Also using } \frac{dr}{dt} = \dot{r}$$

$$\begin{aligned}\text{Equation (3) gives } \vec{v} &= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + \dot{z}\hat{k} \\ \text{which is expression for velocity of particle in cylindrical coordinate.} \quad \dots (5)\end{aligned}$$

Acceleration : The acceleration is given by

$$\begin{aligned}\vec{a} &= \frac{d\vec{v}}{dt} = \frac{d}{dt} \left[\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + \dot{z}\hat{k} \right] \\ \text{or } \vec{a} &= \frac{d}{dt}(\dot{r}\hat{e}_r) + \frac{d}{dt}(r\dot{\theta}\hat{e}_\theta) + \frac{d}{dt}(\dot{z}\hat{k}) \\ &= \ddot{r}\hat{e}_r + \dot{r}\frac{d}{dt}(\hat{e}_r) + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\frac{d}{dt}(\hat{e}_\theta) + \dot{r}\dot{\theta}\hat{e}_\theta + \ddot{z}\hat{k} \quad \dots\dots\dots (6)\end{aligned}$$

$$\text{But } \frac{d}{dt}(\hat{e}_\theta) = \frac{d}{dt}[-\sin\theta\hat{i} + \cos\theta\hat{j}] = -\cos\theta\dot{\theta}\hat{i} - \sin\theta\dot{\theta}\hat{j} = -\dot{\theta}[\cos\theta\hat{i} + \sin\theta\hat{j}] = -\dot{\theta}\hat{e}_r,$$

$$\text{or } \frac{d}{dt}(\hat{e}_r) = \dot{\theta}\hat{e}_\theta \quad \dots\dots\dots (7)$$

$\therefore$ equation(6) becomes $\vec{a} = \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}(-\dot{\theta}\hat{e}_r) + \dot{r}\dot{\theta}\hat{e}_\theta + \ddot{z}\hat{k}$ {using eqⁿ(4), (7)}

$$\text{i.e. } \boxed{\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta + \ddot{z}\hat{k}} \quad \dots\dots\dots (8)$$

which is expression for acceleration of a particle in cylindrical coordinates.

Velocity and acceleration in spherical polar coordinates

The position vector $\vec{r}$ in spherical polar coordinate is given by $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$

$$\begin{aligned}\text{where } x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta\end{aligned}$$

$$\therefore \vec{r} = \hat{i}r\sin\theta\cos\phi + \hat{j}r\sin\theta\sin\phi + \hat{k}r\cos\theta \quad \dots\dots\dots(1)$$

The unit vectors $\hat{e}_r$, $\hat{e}_\theta$ and $\hat{e}_\phi$ along r , θ and ϕ are

$$\hat{e}_r = \frac{\partial \vec{r}}{\partial r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \hat{k}\cos\theta$$

$$\text{as } \left| \frac{\partial \vec{r}}{\partial r} \right| = \sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta} = 1$$

$$\hat{e}_\theta = \frac{\partial \vec{r}}{\partial \theta} = \frac{r\cos\theta\cos\phi\hat{i} + r\cos\theta\sin\phi\hat{j} - \hat{k}r\sin\theta}{\sqrt{r^2(\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta)}} = \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \hat{k}\sin\theta$$

$$\hat{e}_\phi = \frac{\frac{\partial \vec{r}}{\partial \phi}}{\left| \frac{\partial \vec{r}}{\partial \phi} \right|} = \frac{-r \sin \theta \sin \phi \hat{i} + r \sin \theta \cos \phi \hat{j}}{\sqrt{r^2 \sin^2 \theta (\sin^2 \phi + \cos^2 \phi)}} = -\sin \phi \hat{i} + \cos \phi \hat{j}$$

using value of $\hat{e}_r$ in equation (1) we get $\boxed{\vec{r} = r\hat{e}_r}$ (2)

$$\text{So the velocity is given by } \vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{e}_r) = \frac{dr}{dt}\hat{e}_r + r\frac{d}{dt}(\hat{e}_r)$$

$$\text{or } \vec{v} = \dot{r}\hat{e}_r + r\frac{d}{dt}(\hat{e}_r) \quad \text{..... (3)}$$

$$\text{Now } \frac{d}{dt}(\hat{e}_r) = \frac{d}{dt}[\sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \hat{k} \cos \theta]$$

$$\text{or } \frac{d}{dt}(\hat{e}_r) = \cos \theta \dot{\theta} \cos \phi \hat{i} - \sin \theta \sin \phi \dot{\phi} \hat{i} + \cos \theta \dot{\theta} \sin \phi \hat{j} + \sin \theta \cos \phi \dot{\phi} \hat{j} + \hat{k}(-\sin \theta) \dot{\theta}$$

$$\text{or } \frac{d}{dt}(\hat{e}_r) = \dot{\theta}[\cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k}] + \dot{\phi}[\hat{j} \sin \theta \cos \phi - \hat{i} \sin \theta \sin \phi]$$

$$\text{or } \frac{d}{dt}(\hat{e}_r) = \dot{\theta}\hat{e}_\theta + \dot{\phi} \sin \theta \hat{e}_\phi \quad \text{..... (4) \{using values of } \hat{e}_\theta \text{ and } \hat{e}_\phi \text{\}}$$

So using equation (4) in equation (3) we get

$$\vec{v} = \dot{r}\hat{e}_r + r(\dot{\theta}\hat{e}_\theta + \dot{\phi} \sin \theta \hat{e}_\phi)$$

$$\text{or } \boxed{\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\sin \theta \dot{\phi}\hat{e}_\phi} \quad \text{..... (5) which is the expression for velocity of particle in spherical polar coordinates}$$

Expression For Acceleration

$$\text{The acceleration is given by } \vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}[\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\sin \theta \dot{\phi}\hat{e}_\phi]$$

$$\text{or } \vec{a} = \frac{d}{dt}(\dot{r}\hat{e}_r) + \frac{d}{dt}(r\dot{\theta}\hat{e}_\theta) + \frac{d}{dt}(r\sin \theta \dot{\phi}\hat{e}_\phi)$$

$$\text{or } \vec{a} = \ddot{r}\hat{e}_r + \dot{r}\frac{d}{dt}(\hat{e}_r) + r\ddot{\theta}\hat{e}_\theta + \dot{r}\dot{\theta}\hat{e}_\theta + r\dot{\theta}\frac{d}{dt}(\hat{e}_\theta) + \dot{r}\sin \theta \dot{\phi}\hat{e}_\phi + r\cos \theta \dot{\theta}\dot{\phi}\hat{e}_\phi + r\sin \theta \dot{\phi}\frac{d}{dt}(\hat{e}_\phi) + r\sin \theta \hat{e}_\phi \ddot{\phi}$$

.....(6)

$$\begin{aligned}\text{Now } \frac{d}{dt}(\hat{e}_\theta) &= \frac{d}{dt}[\cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \hat{k}\sin\theta] \\ &= -\sin\theta\dot{\theta}\cos\phi\hat{i} - \cos\theta\sin\phi\dot{\phi}\hat{i} - \sin\theta\dot{\theta}\sin\phi\hat{j} + \dot{\phi}\cos\theta\cos\phi\hat{j} - \hat{k}\cos\theta\dot{\theta} \\ \text{or } \frac{d}{dt}(\hat{e}_\theta) &= -\dot{\theta}[\sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}] + \dot{\phi}\cos\theta(-\sin\phi\hat{i} + \cos\phi\hat{j}) \\ &= -\dot{\theta}\hat{e}_r + \dot{\phi}\cos\theta\hat{e}_\phi \quad \dots\dots\dots (7) \text{ \{using values of } \hat{e}_r \text{ and } \hat{e}_\phi \text{\}}\end{aligned}$$

$$\begin{aligned}\text{Also } \frac{d}{dt}(\hat{e}_\phi) &= \frac{d}{dt}(-\sin\phi\hat{i} + \cos\phi\hat{j}) \quad \dots\dots\dots (8) \\ &= -\cos\phi\dot{\phi}\hat{i} - \sin\phi\dot{\phi}\hat{j}\end{aligned}$$

So using the values of $\frac{d}{dt}(\hat{e}_r)$, $\frac{d}{dt}(\hat{e}_\theta)$ and $\frac{d}{dt}(\hat{e}_\phi)$ in eqⁿ (6) we get

$$\begin{aligned}\vec{a} &= \ddot{r}\hat{e}_r + \dot{r}(\dot{\theta}\hat{e}_\theta + \dot{\phi}\sin\theta\hat{e}_\phi) + r\ddot{\theta}\hat{e}_\theta + \dot{r}\dot{\theta}\hat{e}_\theta + r\dot{\theta}(-\dot{\theta}\hat{e}_r + \dot{\phi}\cos\theta\hat{e}_\phi) \\ &\quad + \dot{r}\sin\theta\dot{\phi}\hat{e}_\phi + r\cos\theta\dot{\theta}\dot{\phi}\hat{e}_\phi + r\sin\theta\dot{\phi}(-\cos\phi\dot{\phi}\hat{i} - \sin\phi\dot{\phi}\hat{j}) + r\sin\theta\hat{e}_\phi\ddot{\phi} \\ \Rightarrow \vec{a} &= \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\dot{\phi}\sin\theta\hat{e}_\phi + r\ddot{\theta}\hat{e}_\theta + \dot{r}\dot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r + r\dot{\theta}\dot{\phi}\cos\theta\hat{e}_\phi \\ &\quad + r\sin\theta\hat{e}_\phi\ddot{\phi} + \dot{r}\sin\theta\dot{\phi}\hat{e}_\phi + r\cos\theta\dot{\theta}\dot{\phi}\hat{e}_\phi - \dot{\phi}^2 r\sin\theta(\cos\phi\hat{i} + \sin\phi\hat{j}) \\ \Rightarrow \vec{a} &= \ddot{r}\hat{e}_r - r\dot{\theta}^2\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + \dot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\dot{\phi}\sin\theta\hat{e}_\phi + r\dot{\theta}\dot{\phi}\cos\theta\hat{e}_\phi \\ &\quad + \dot{r}\sin\theta\dot{\phi}\hat{e}_\phi + r\cos\theta\dot{\theta}\dot{\phi}\hat{e}_\phi - \dot{\phi}^2 r\sin\theta(\sin\theta\hat{e}_r + \cos\theta\hat{e}_\theta) \\ &\quad \quad \quad \{\because \cos\phi\hat{i} + \sin\phi\hat{j} = \sin\theta\hat{e}_r + \cos\theta\hat{e}_\theta\} \\ \Rightarrow \vec{a} &= \left(\ddot{r} - r\dot{\theta}^2 - \dot{\phi}^2 r\sin^2\theta\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2\right)\hat{e}_\theta \\ &\quad + \left(r\sin\theta\ddot{\phi} + 2\dot{r}\dot{\phi}\sin\theta + 2r\cos\theta\dot{\theta}\dot{\phi}\right)\hat{e}_\phi \quad \dots\dots\dots (9)\end{aligned}$$

which is the expression for acceleration of a particle in spherical polar coordinates.